Computations of unitary groups in characteristic 2

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Contents

Chapter 1

Introduction

1.1   The package Blocks

The package Blocks is a GAP4 package containing all the code developped to address J.-P. Serre's question regarding the group of components of the unitary group of a group algebra (or, more generally, of an involutive algebra) in characteristic 2. The unitary group of an involutive algebra A is the affine algebraic group defined by the equation $S(a)a = 1$, where S denotes the involution of A. This question appeared naturally in [Ser14].

The specific question, communicated to our workgroup by Gunter Malle, was about the group algebra of A_5. The included programs are generally only feasible for small group orders. Together with further theoretical arguments the answer to the question for A_5 is now in computational reach: The group of components of the unitary group of the group ring $\overline{\mathbb{F}}_2 A_5$ is trivial. A modern computer took 19 hours to produce this answer.

A considerable part of the package now consists of algorithms for standard block theoretical computations. These algorithms, written for further refined computer-assisted experiments, were suprsingly missing in GAP4. The package Blocks is based on the homalg project [Thpa13], [BLH11], [BLH12] which is entirely written in GAP4 [GAP12], [BL98].

The downloadable archive and the online documentation of the GAP4 package Blocks can be found under (http://homalg.math.rwth-aachen.de/~barakat/homalg-project/Blocks). The source code can be viewed under (http://github.com/homalg-project/Blocks). Blocks requires new versions of several packages of the homalg project which have been released with GAP 4.7.7. It also requires the (not released yet) package ctblocks (v0.9.1) of Thomas Breuer.

The online server (see http://homalg.math.rwth-aachen.de/) offers a free web-based access to GAP4 and Singular with the newest packages of the homalg project preinstalled. Only a modern browser is required. This access can either be used for further computer-assisted experiments or simply to reproduce and verify any computation shown in this document. For this one can simply cut and paste the input lines of any example shown below (in the correct order); the GAP prompt gap> will be automatically suppressed during pasting.

Chapter 2

The group of components of the unitary group of a group ring

2.1 Notation

Let A be a k-algebra with k-linear involution S over a field k. The unitary group $U(A)$ is the smooth linear algebraic group defined by the equation $S(a)a = 1$. Let I be an S-stable two-sided ideal $I \trianglelefteq A$. The natural epimorphism $A \to A/I$ induces a morphism

$$\varphi_{A/I} : U(A) \to U(A/I)$$

of algebraic groups. This morphism is in characteristic 2 generally *not* onto. Denote by

$$U(A,I) := U(A) \cap (1+I) \quad \text{and} \quad U(A,A/I) := U(A)/U(A,I)$$

the kernel and coimage of $\varphi_{A/I}$, respectively. Given another S-stable ideal $J \subset I$ define

$$U(A,I/J) := U(A,I)/U(A,J).$$

The subfactor group $U(A,I/J)$ is the coimage of the composed morphism

$$\varphi_{I/J} : U(A,I) \hookrightarrow U(A) \overset{\varphi_{A/J}}{\to} U(A/J) \hookrightarrow A/J.$$

When computing we will rather consider its corestriction to the affine subspace $(1+I)/J \subset A/J$

$$\varphi_{I/J} : U(A,I) \to (1+I)/J.$$

Since we can compute (Zariski closures of) images of affine morphisms we can compute $U(A,I/J)$, up to a natural isomorphism.

Let $\mathfrak{r}$ be the radical of A. We will call the factor group $U(A,A/\mathfrak{r})$ the *upper part*, the subfactor group $U(A,\mathfrak{r}/\mathfrak{r}^2)$ the *middle part*, and the subgroup $U(A,\mathfrak{r}^2)$ the *lower part* of the unitary group $U(A)$.

2.2 The algorithm used to compute the closed image

The defining ideal of the closure of the image of a morphism of affine varieties is the kernel of the associated morphism of coordinate rings. Computing the kernel leads to an elimination problem in a polynomial ring.

Let $k[x_1, \ldots, x_i]$ be the coordinate ring of the affine variety $1 + I$ and $k[y_1, \ldots, y_d]$ the coordinate ring of $(1 + I)/J$. After an appropriate choice of coordinates (adapted to the inclusion $I \supset J$) the projection $\pi : 1 + I \twoheadrightarrow (1+I)/J$ amounts to mapping $y_k \mapsto x_k$ for all $k = 1, \ldots, d := \dim_k I/J$. Finally, the associated morphism of coordinate rings $\Gamma(\mathscr{O}_{U(A,I)}) = k[x_1, \ldots, x_i]/M \xleftarrow{\varphi_{I/J}} k[y_1, \ldots, y_d]$ is given by mapping $y_k \mapsto x_k + M$ for all $k = 1, \ldots, d$, where M is the defining ideal of $U(A,I) \subset 1 + I$. For an example see Subsection 2.10.1. For details on the A_5 computation see the comments in subsection 2.12.2.

Hence, computing the kernel reduces to intersecting the ideal M with the subring $k[x_1, \ldots, x_d] \leq k[x_1, \ldots, x_i]$ (identified via π with $k[y_1, \ldots, y_d]$), or, equivalently, eliminating the indeterminates $x_{d+1}, \ldots, x_i$. This can be done by computing a Gröbner basis of M with respect to the block elimination ordering $((x_1, \ldots, x_d), (x_{d+1}, \ldots, x_i))$.

Computations with elimination orderings are known to be expensive. The choice of adapted coordinates has an enormous impact on the run-time. Depending on this choice the computation of a kernel via elimination might range from few hours to several days. It might very well be possible that a good choice of coordinates could make the elimination doable by hand.

2.3   Some experimental facts on the unitary groups of 2-blocks

Let G be a finite group and $A = \overline{\mathbb{F}}_2 G$ its group algebra with radical $\mathfrak{r}$. By G° we denote the kernel, and by E_G the cokernel of the composition $G \hookrightarrow U(A) \twoheadrightarrow \pi_0(U(A))$ (cf. [Ser05, Section 2]). G° is the characteristic subgroup of G generated by the squares and the involutions. Denote by $B_0, B_1, \ldots, B_n$ the involution-stable 2-blocks of A with radicals $\mathfrak{r}_0, \mathfrak{r}_1, \ldots, \mathfrak{r}_n$, respectively, where B_0 is the principal 2-block.

The following statements are true for all computed examples and imply that

$$E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^{n} \pi_0(U(B_j, B_j/\mathfrak{r}_j)).$$

Note that the product does not involve the principal block. A general validity of this isomorphism would imply that E_G is trivial for all finite 2-groups (the latter statement would also follow from weaker statements).

1. $U(B_j, \mathfrak{r}_j^i/\mathfrak{r}_j^{i+1})$ is an affine space (hence connected) for all j and all $i \geq 2$.

2. The natural morphism $G/G^\circ \to \pi_0(U(B_0, B_0/\mathfrak{r}_0^2))$ is an isomorphism.

    In particular if $U(B_0, B_0/\mathfrak{r}_0)$ is connected (which is not always the case, see 2.11.1 and 2.11.2) then $G/G^\circ \cong \pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2))$.

3. $U(B_j, \mathfrak{r}_j/\mathfrak{r}_j^2)$ is connected for all $j > 0$.

4. The natural morphism $G/G^\circ \to \pi_0(U(B_j, B_j/\mathfrak{r}_j))$ is surjective (for all $j > 0$).

2.4   The paper of Bovdi and Rosa

The results of the paper [BR00] determine the ζ-function of the unitary group in characteristic 2, and imply that $E_G = 0$ for some classes of non-Abelian 2-groups. They denote the group $U_G(k)$ of k-rational points of U_G by $V_*(kG)$. Using the formula $\dim U_G = \frac{1}{2}(|G| + |G[2]|) - 1$ and the order

of G/G° one can unify the formulation of all results in loc. cit.. This raises the question of whether the unified formulation of these results is valid for all finite 2-groups. An affirmative answer would imply that $E_G = 0$ for all finite 2-groups.

THEOREM (Unified formulation of [BR00, Theorems 1 and 2]). For all 2-groups listed below

$$|V_*(kG)| = |U_G(k)| = |G/G^\circ| \cdot |k|^{\dim U_G}.$$

Here G is one of the following 2-groups:

Thm 1:
   Let H is an extraspecial 2-group of order 2^{2n+1}, then

   1. $G = H$, or

   2. G is an central product H with a cyclic group of order 4.

Thm 2:
   G has an Abelian subgroup A of index 2 and an element b which inverts each element of A (i.e., $a^b = a^{-1}$), where

   1. b is of order 2, or

   2. b is of order 4.

In the first three cases of the theorem $|G/G^\circ| = 1$. In the last case the equality is stated with $2 \cdot |A^2[2]|$ instead of $|G/G^\circ|$. ($A^2 := \{a^2 \mid a \in A\}$ and $A[2] := \{a \in A \mid \text{ord } a \mid 2\}$. Of course, these two subsets are subgroups when A is Abelian, namely the image and kernel of the squaring homomorphism.) To see that the above two numbers coincide we may argue as follows. From ord $b = 4$ we conclude that $G/G^\circ = C_2 \times A/A^\circ$. Since A is Abelian we can write $A^\circ = A^2 A[2]$. We claim that $A/A^\circ \cong A^2[2]$. It suffices to check the previous isomorphism for cyclic 2-groups $A = C_{2^n}$, $n \geq 1$. Indeed $C_{2^n}^2 = C_{2^{n-1}}$ and $C_{2^n}[2] = C_2$. Hence,

$$C_{2^n}/C_{2^n}^\circ \cong C_{2^n}^2[2] \cong C_2 \text{ for } n > 1, \text{ otherwise trivial}.$$

2.5 Description of the input commands

- The input for the computer is colored red and preceded by the blue prompt "gap>". A semicolon indicates the end of a command. Two consecutive semicolons suppress the output of the preceding command.

- The output of the computer is black.

- `LoadPackage( "Blocks" )` loads the GAP package Blocks.

- `StructureDescription(` G `)` prints one of the standard names of the isomorphism type of the group G.

- `SerreFactorGroup` computes G/G°.

- `k := HomalgRingOfIntegersInSingular( p^d )` triggers the Gröbner basis oracle Singular and defines there the ground field $k := GF(p^d)$.

- `SetCoefficientsRingForPolynomialAlgebra( kG, k )` tells the computer to perform all Gröbner basis related computations for kG in the oracle Singular over k.

- `DefiningIdealOfLowerPartOfUnitaryGroup( Bj )` computes the defining ideal of the subgroup $U(B_j, \mathfrak{r}_j^2)$.

- `IsLowerPartExtensionOfAffineSpaces( Bj )` checks statement 1 in Section 2.3.

- `DefiningIdealOfMiddlePartOfUnitaryGroup( Bj )` computes the defining ideal of the subfactor group $U(B_j, \mathfrak{r}_j/\mathfrak{r}_j^2)$, which is *not* necessarily connected.

- `DefiningIdealOfUpperPartOfUnitaryGroup( Bj )` computes the defining ideal of the factor group $U(B_j, B_j/\mathfrak{r}_j)$, which is *not* necessarily connected.

- I verified statement 4 for all computed examples with nontrivial E_G. This verification is still not included below.

- `CentralNonPrincipalIdempotentsOfInvolutiveAlgebra( kG )` computes the list of central non-principal primitive idempotents of the group algebra kG, viewed as an involutive algebra. This means that the list contains all real block idempotents, with the principal idempotent excluded, and for each complex block idempotent e_i the sum with its conjugate idempotent `Involution`(e_i).

- `Perform( L, command )` performs `command` (e.g., `Display`) on each entry of the list L.

- `RadicalDecomposition( I )` computes the list of associated primes of the ideal `I`. Even if the ideal `I` is radical (which is normally the case here) the computation of `RadicalDecomposition( I )` is generally faster than `PrimaryDecomposition( I )` since the computer skips the search for primary non-prime ideals.

2.6 Comments on the computer output

- The matrix to which outputs like "`An ideal generated by the entry of the above matrix`" or "`An ideal generated by the 3 entries of the above matrix`" is referring is not printed with surrounding brackets.

- In the output of `CentralNonPrincipalIdempotentsOfGroupAlgebra( kG )` the symbol `Z(2^d)^n` stands for the *n*-th power of a primitive element `Z(2^d)` of $GF(2^d)$. In particular, `Z(2)^0` stands for the (common) one in all finite fields $GF(2^d)$'s. This is a convention of the computer algebra system GAP which I cannot influence. When ideals are displayed the symbol `Z2_d^n` corresponds to `Z(2^d)^n`.

- The adjective "torsion-free" in outputs like "`A torsion-free ideal ...`" is of course clear from the context. We might suppress such superfluous adjectives in the future.

2.7   Order 6

2.7.1   S3

```
——————————————————— Example ———————————————————
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 6, 1 );; StructureDescription( G );; G;
S3
gap> StructureDescription( SerreFactorGroup( G ) );
"1"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 2 generators>
gap> DimensionOfUnitaryGroup( kG );
4
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
——————————————————— Example ———————————————————
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 2 generators>,
  (dimension 2)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i/\mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
——————————————————— Example ———————————————————
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A zero vector subspace>
gap> AffineDimension( L0 );
0
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:

```
——————————————————— Example ———————————————————
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A zero ideal>
gap> AffineDimension( M0 );
1
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
——————————————————— Example ———————————————————
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( I0 );
0
gap> Dec0 := RadicalDecomposition( I0 );
[ <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( Dec0, Display );
y1+1

An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block:

```
─────────────────────────── Example ───────────────────────────
gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
[ <two-sided ideal in <algebra-with-one of dimension 6 over GF(2)>,
    (dimension 4)> ]
gap> B1 := B[1];
<two-sided ideal in <algebra-with-one of dimension 6 over GF(2)>,
  (dimension 4)>
gap> IsSpecial( B1 );
false
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i / \mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
─────────────────────────── Example ───────────────────────────
gap> DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
<A zero vector subspace>
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1 / \mathfrak{r}_1^2)$ is connected:

```
─────────────────────────── Example ───────────────────────────
gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
<A zero vector subspace>
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^{n} \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
─────────────────────────── Example ───────────────────────────
gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( I1 );
3
gap> AffineDegree( I1 );
2
gap> Display( I1 );
y1^2+y2^2+y1*y3+y3^2+y2*y4+y4^2+1

An ideal generated by the entry of the above matrix
```

2.8 Order 12

2.8.1 C3 : C4

This is the smallest group G with nontrivial E_G and the unique of order 12.

```
─────────────────────────── Example ───────────────────────────
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 12, 1 );; StructureDescription( G );; G;
C3 : C4
gap> StructureDescription( SerreFactorGroup( G ) );
"C2"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 3 generators>
gap> DimensionOfUnitaryGroup( kG );
6
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
──────────────────── Example ────────────────────
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 3 generators>,
  (dimension 4)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i/\mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
──────────────────── Example ────────────────────
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A zero ideal>
gap> AffineDimension( L0 );
2
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:

```
──────────────────── Example ────────────────────
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M0 );
0
gap> m0 := RadicalDecomposition( M0 );
[ <A principal torsion-free ideal given by a cyclic generator>,
  <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( m0, Display );
y1+1

An ideal generated by the entry of the above matrix
y1

An ideal generated by the entry of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
──────────────────── Example ────────────────────
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( I0 );
0
gap> Dec0 := RadicalDecomposition( I0 );
[ <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( Dec0, Display );
y1+1

An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block:

```
──────────────────── Example ────────────────────
gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
[ <two-sided ideal in <algebra-with-one of dimension 12 over GF(2)>,
    (dimension 8)> ]
gap> B1 := B[1];
<two-sided ideal in <algebra-with-one of dimension 12 over GF(2)>,
  (dimension 8)>
gap> IsSpecial( B1 );
true
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i/\mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
──────────────────────────── Example ────────────────────────────
 gap> DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
 <A zero vector subspace>
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1/\mathfrak{r}_1^2)$ is connected:

```
──────────────────────────── Example ────────────────────────────
 gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
 <A principal torsion-free ideal given by a cyclic generator>
 gap> AffineDimension( M1 );
 3
 gap> Display( M1 );
 y1

 An ideal generated by the entry of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^{n} \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
──────────────────────────── Example ────────────────────────────
 gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
 <A torsion-free ideal given by 7 generators>
 gap> AffineDimension( I1 );
 1
 gap> Dec1 := RadicalDecomposition( I1 );
 [ <A torsion-free ideal given by 3 generators>,
   <A torsion-free ideal given by 3 generators> ]
 gap> Perform( Dec1, Display );
 y4,
 y2,
 y1^2+y1*y3+y3^2+1

 An ideal generated by the 3 entries of the above matrix
 y3,
 y1,
 y2^2+y2*y4+y4^2+1

 An ideal generated by the 3 entries of the above matrix
```

The ζ-function (in characteristic 2) of the 1-dimensional degree 4 factor group $U(kG, kG/\mathfrak{r})$ is

$$\zeta_{U(kG, kG/\mathfrak{r})}(t) = \frac{(1+t)^2}{(1-2t)^2},$$

where $\mathfrak{r}$ is the radical of kG. As shown above, the number of connected components of $U(kG, kG/\mathfrak{r})$ equals $|E_G| = 2$.

2.9 Order 20

2.9.1 C5 : C4

This is the second smallest group G with nontrivial E_G and the unique of order 20.

```
──────────────── Example ────────────────
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 20, 1 );;  StructureDescription( G );;  G;
C5 : C4
gap> StructureDescription( SerreFactorGroup( G ) );
"C2"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2^2), with 3 generators>
gap> DimensionOfUnitaryGroup( kG );
10
gap> d := DegreeOverPrimeField( LeftActingDomain( kG ) );
2
gap> k := HomalgRingOfIntegersInSingular( 2, d );
GF(2^2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
──────────────── Example ────────────────
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2^2), with 3 generators>,
  (dimension 4)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i/\mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
──────────────── Example ────────────────
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A zero ideal>
gap> AffineDimension( L0 );
2
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:

```
──────────────── Example ────────────────
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M0 );
0
gap> m0 := RadicalDecomposition( M0 );
[ <A principal torsion-free ideal given by a cyclic generator>,
  <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( m0, Display );
y1+1

An ideal generated by the entry of the above matrix
y1

An ideal generated by the entry of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
──────────────── Example ────────────────
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
```

```
gap> AffineDimension( I0 );
0
gap> Dec0 := RadicalDecomposition( I0 );
[ <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( Dec0, Display );
y1+1


An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block B_1:

```
───────────────────────────── Example ─────────────────────────────
gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
[ <two-sided ideal in <algebra-with-one of dimension 20 over GF(2^2)>,
   (dimension 8)>,
  <two-sided ideal in <algebra-with-one of dimension 20 over GF(2^2)>,
   (dimension 8)> ]
gap> B1 := B[1];
<two-sided ideal in <algebra-with-one of dimension 20 over GF(2^2)>,
 (dimension 8)>
gap> IsSpecial( B1 );
true
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i / \mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
───────────────────────────── Example ─────────────────────────────
gap> DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
<A zero vector subspace>
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1 / \mathfrak{r}_1^2)$ is connected:

```
───────────────────────────── Example ─────────────────────────────
gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M1 );
3
gap> Display( M1 );
y1


An ideal generated by the entry of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^n \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2 \times C_2$:

```
───────────────────────────── Example ─────────────────────────────
gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 7 generators>
gap> AffineDimension( I1 );
1
gap> Dec1 := RadicalDecomposition( I1 );
[ <A torsion-free ideal given by 3 generators>,
  <A torsion-free ideal given by 3 generators> ]
gap> Perform( Dec1, Display );
y4,
y2,
y1^2+(Z2_2)*y1*y3+y3^2+(Z2_2+1)
```

```
An ideal generated by the 3 entries of the above matrix
y3,
y1,
y2^2+(Z2_2)*y2*y4+y4^2+(Z2_2+1)


An ideal generated by the 3 entries of the above matrix
```

Now we consider the non-principal block B_2:

```
──────────────── Example ────────────────
gap> B2 := B[2];
<two-sided ideal in <algebra-with-one of dimension 20 over GF(2^2)>,
 (dimension 8)>
gap> IsSpecial( B2 );
true
```

Verify statement 1 for $j = 2$, i.e., $U(B_2, \mathfrak{r}_2^i/\mathfrak{r}_2^{i+1})$ is connected for all $i \geq 2$:

```
──────────────── Example ────────────────
gap> DefiningIdealOfLowerPartOfUnitaryGroup( B2 );
<A zero vector subspace>
```

Verify statement 3, i.e., $U(B_2, \mathfrak{r}_2/\mathfrak{r}_2^2)$ is connected:

```
──────────────── Example ────────────────
gap> M2 := DefiningIdealOfMiddlePartOfUnitaryGroup( B2 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M2 );
3
gap> Display( M2 );
y1

An ideal generated by the entry of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^{n} \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2 \times C_2$:

```
──────────────── Example ────────────────
gap> I2 := DefiningIdealOfUpperPartOfUnitaryGroup( B2 );
<A torsion-free ideal given by 7 generators>
gap> AffineDimension( I2 );
1
gap> Dec2 := RadicalDecomposition( I2 );
[ <A torsion-free ideal given by 3 generators>,
  <A torsion-free ideal given by 3 generators> ]
gap> Perform( Dec2, Display );
y4,
y2,
y1^2+(Z2_2+1)*y1*y3+y3^2+(Z2_2)

An ideal generated by the 3 entries of the above matrix
y3,
y1,
y2^2+(Z2_2+1)*y2*y4+y4^2+(Z2_2)


An ideal generated by the 3 entries of the above matrix
```

2.10   Order 24

The first three groups have nontrivial E_G.

2.10.1   C2 x (C3 : C4)

```
———————————————————————— Example ————————————————————————
 gap> LoadPackage( "Blocks" );
 true
 gap> G := SmallGroup( 24, 7 );;  StructureDescription( G );;  G;
 C2 x (C3 : C4)
 gap> StructureDescription( SerreFactorGroup( G ) );
 "C2"
 gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
 <algebra-with-one over GF(2), with 4 generators>
 gap> DimensionOfUnitaryGroup( kG );
 13
 gap> k := HomalgRingOfIntegersInSingular( 2 );
 GF(2)
 gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
———————————————————————— Example ————————————————————————
 gap> B0 := PrincipalBlock( kG );
 <two-sided ideal in <algebra-with-one over GF(2), with 4 generators>,
   (dimension 8)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
———————————————————————— Example ————————————————————————
 gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
 <A principal torsion-free ideal given by a cyclic generator>
 gap> AffineDimension( L0 );
 4
 gap> Display( L0 );
 x2+x4

 An ideal generated by the entry of the above matrix
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:

```
———————————————————————— Example ————————————————————————
 gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
 <A principal torsion-free ideal given by a cyclic generator>
 gap> AffineDimension( M0 );
 1
 gap> m0 := RadicalDecomposition( M0 );
 [ <A principal torsion-free ideal given by a cyclic generator>,
   <A principal torsion-free ideal given by a cyclic generator> ]
 gap> Perform( m0, Display );
 y1+y2+1

 An ideal generated by the entry of the above matrix
 y1+y2
```

```
   An ideal generated by the entry of the above matrix
```

Now we show the morphism $\varphi_{\mathfrak{r}_0/\mathfrak{r}_0^2}$ of which the defining ideal M0 is the kernel:

```
─────────────────────── Example ───────────────────────
gap> phi := DefiningMorphismOfMiddlePartOfUnitaryGroup( B0 );
<A homomorphism of rings>
gap> Display( phi );
GF(2)[x1,x2,x3,x4,x5,x6,x7]/(x1*x2+x2^2+x1+x2+x3+x4+x5+x6, x1^2+x1*x2+x3+x4+x5+x6)
  ^
  |
[ x1, x2 ]
  |
  |
GF(2)[y1,y2]
gap> N0 :=KernelSubobject( phi );
<A principal torsion-free ideal given by a cyclic generator>
gap> N0 = M0;
true
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
─────────────────────── Example ───────────────────────
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( I0 );
0
gap> Dec0 := RadicalDecomposition( I0 );
[ <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( Dec0, Display );
y1+1

An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block:

```
─────────────────────── Example ───────────────────────
gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
[ <two-sided ideal in <algebra-with-one of dimension 24 over GF(2)>,
   (dimension 16)> ]
gap> B1 := B[1];
<two-sided ideal in <algebra-with-one of dimension 24 over GF(2)>,
 (dimension 16)>
gap> IsSpecial( B1 );
true
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i/\mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
─────────────────────── Example ───────────────────────
gap> L1 := DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( L1 );
3
gap> Display( L1 );
```

```
    x3

    An ideal generated by the entry of the above matrix
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1/\mathfrak{r}_1^2)$ is connected:

```
───────────────────────── Example ─────────────────────────
 gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
 <A torsion-free ideal given by 4 generators>
 gap> AffineDimension( M1 );
 4
 gap> Display( M1 );
 y6+y8,
 y3,
 y2+y5,
 y1

 An ideal generated by the 4 entries of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^n \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
───────────────────────── Example ─────────────────────────
 gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
 <A torsion-free ideal given by 7 generators>
 gap> AffineDimension( I1 );
 1
 gap> Dec1 := RadicalDecomposition( I1 );
 [ <A torsion-free ideal given by 3 generators>,
   <A torsion-free ideal given by 3 generators> ]
 gap> Perform( Dec1, Display );
 y4,
 y2,
 y1^2+y1*y3+y3^2+1

 An ideal generated by the 3 entries of the above matrix
 y3,
 y1,
 y2^2+y2*y4+y4^2+1

 An ideal generated by the 3 entries of the above matrix
```

2.10.2   C3 : C8

```
───────────────────────── Example ─────────────────────────
 gap> LoadPackage( "Blocks" );
 true
 gap> G := SmallGroup( 24, 1 );; StructureDescription( G );; G;
 C3 : C8
 gap> StructureDescription( SerreFactorGroup( G ) );
 "C2"
 gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
 <algebra-with-one over GF(2), with 4 generators>
 gap> DimensionOfUnitaryGroup( kG );
 12
```

```
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
———————— Example ————————
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 4 generators>,
  (dimension 8)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i/\mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
———————— Example ————————
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 2 generators>
gap> AffineDimension( L0 );
4
gap> Display( L0 );
x1*x2+x2*x3+x1*x4+x3*x4+x2*x5+x4*x5+x1*x6+x3*x6+x5*x6+x2+x6,
x1^2+x2^2+x3^2+x4^2+x5^2+x6^2+x1+x3+x5

An ideal generated by the 2 entries of the above matrix
gap> IsLowerPartExtensionOfAffineSpaces( B0 );
true
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:

```
———————— Example ————————
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M0 );
0
gap> m0 := RadicalDecomposition( M0 );
[ <A principal torsion-free ideal given by a cyclic generator>,
  <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( m0, Display );
y1+1

An ideal generated by the entry of the above matrix
y1

An ideal generated by the entry of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
———————— Example ————————
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( I0 );
0
gap> Dec0 := RadicalDecomposition( I0 );
[ <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( Dec0, Display );
y1+1

An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block:

```
────────────── Example ──────────────
gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
[ <two-sided ideal in <algebra-with-one of dimension 24 over GF(2)>,
   (dimension 16)> ]
gap> B1 := B[1];
<two-sided ideal in <algebra-with-one of dimension 24 over GF(2)>,
 (dimension 16)>
gap> IsSpecial( B1 );
true
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i / \mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
────────────── Example ──────────────
gap> L1 := DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 4 generators>
gap> AffineDimension( L1 );
4
gap> Display( L1 );
x5,
x2+x4,
x7,
x6+x8

An ideal generated by the 4 entries of the above matrix
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1 / \mathfrak{r}_1^2)$ is connected:

```
────────────── Example ──────────────
gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M1 );
3
gap> Display( M1 );
y1

An ideal generated by the entry of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^n \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
────────────── Example ──────────────
gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 7 generators>
gap> AffineDimension( I1 );
1
gap> Dec1 := RadicalDecomposition( I1 );
[ <A torsion-free ideal given by 3 generators>,
  <A torsion-free ideal given by 3 generators> ]
gap> Perform( Dec1, Display );
y4,
y2,
y1^2+y1*y3+y3^2+1

An ideal generated by the 3 entries of the above matrix
y3,
```

```
  y1,
  y2^2+y2*y4+y4^2+1

  An ideal generated by the 3 entries of the above matrix
```

2.10.3   C3 : Q8

```
────────────────── Example ──────────────────
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 24, 4 );; StructureDescription( G );; G;
C3 : Q8
gap> StructureDescription( SerreFactorGroup( G ) );
"C2 x C2"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 4 generators>
gap> DimensionOfUnitaryGroup( kG );
12
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
────────────────── Example ──────────────────
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 4 generators>,
  (dimension 8)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
────────────────── Example ──────────────────
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( L0 );
4
gap> Display( L0 );
x2+x4

An ideal generated by the entry of the above matrix
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong C_2 \times C_2 \cong G/G^\circ$:

```
────────────────── Example ──────────────────
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 2 generators>
gap> AffineDimension( M0 );
0
gap> m0 := RadicalDecomposition( M0 );
[ <A torsion-free ideal given by 2 generators>,
  <A torsion-free ideal given by 2 generators>,
  <A torsion-free ideal given by 2 generators>,
  <A torsion-free ideal given by 2 generators> ]
gap> Perform( m0, Display );
y2,
```

```
  y1+1

An ideal generated by the 2 entries of the above matrix
  y2+1,
  y1+1

An ideal generated by the 2 entries of the above matrix
  y2,
  y1

An ideal generated by the 2 entries of the above matrix
  y2+1,
  y1

An ideal generated by the 2 entries of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
────────────────────── Example ──────────────────────
 gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
 <A principal torsion-free ideal given by a cyclic generator>
 gap> AffineDimension( I0 );
 0
 gap> Dec0 := RadicalDecomposition( I0 );
 [ <A principal torsion-free ideal given by a cyclic generator> ]
 gap> Perform( Dec0, Display );
 y1+1

 An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block:

```
────────────────────── Example ──────────────────────
 gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
 [ <two-sided ideal in <algebra-with-one of dimension 24 over GF(2)>,
    (dimension 16)> ]
 gap> B1 := B[1];
 <two-sided ideal in <algebra-with-one of dimension 24 over GF(2)>,
  (dimension 16)>
 gap> IsSpecial( B1 );
 true
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i/\mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
────────────────────── Example ──────────────────────
 gap> L1 := DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
 <A torsion-free ideal given by 2 generators>
 gap> AffineDimension( L1 );
 6
 gap> Display( L1 );
 x5,
 x7

 An ideal generated by the 2 entries of the above matrix
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1/\mathfrak{r}_1^2)$ is connected:

```
──────────────────── Example ────────────────────
gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 3 generators>
gap> AffineDimension( M1 );
1
gap> Display( M1 );
y4,
y2,
y1

An ideal generated by the 3 entries of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^{n} \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
──────────────────── Example ────────────────────
gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 7 generators>
gap> AffineDimension( I1 );
1
gap> Dec1 := RadicalDecomposition( I1 );
[ <A torsion-free ideal given by 3 generators>,
  <A torsion-free ideal given by 3 generators> ]
gap> Perform( Dec1, Display );
y4,
y2,
y1^2+y1*y3+y3^2+1

An ideal generated by the 3 entries of the above matrix
y3,
y1,
y2^2+y2*y4+y4^2+1

An ideal generated by the 3 entries of the above matrix
```

2.10.4 C2 x A4

```
──────────────────── Example ────────────────────
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 24, 13 );; StructureDescription( G );; G;
C2 x A4
gap> StructureDescription( SerreFactorGroup( G ) );
"1"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 4 generators>
gap> DimensionOfUnitaryGroup( kG );
15
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

There are no non-principal blocks, so we only consider the principal block:

──────────────────────────────── Example ────────────────────────────────
```
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 4 generators>,
  (dimension 24)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

──────────────────────────────── Example ────────────────────────────────
```
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 6 generators>
gap> AffineDimension( L0 );
8
gap> Display( L0 );
x2+x7,
x4+x7+x9,
x6+x8+x9,
x2+x4+x6+x8,
x2+x4+x9,
x10+x12


An ideal generated by the 6 entries of the above matrix
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:

──────────────────────────────── Example ────────────────────────────────
```
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 3 generators>
gap> AffineDimension( M0 );
6
gap> Display( M0 );
y6+y7+y9,
y5+y7+y8+y9,
y3+y8


An ideal generated by the 3 entries of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0 / \mathfrak{r}_0)$ is connected:

──────────────────────────────── Example ────────────────────────────────
```
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 3 generators>
gap> AffineDimension( I0 );
1
gap> Dec0 := RadicalDecomposition( I0 );
[ <A torsion-free ideal given by 2 generators> ]
gap> AffineDegree( Dec0[1] );
2
gap> Perform( Dec0, Display );
y1+y2+y3+1,
y2^2+y2*y3+y3^2+y2+y3


An ideal generated by the 2 entries of the above matrix
```

2.10.5 S4

The following computations show that E_{S_4} is trivial.

```
                            ──────── Example ────────
  gap> LoadPackage( "Blocks" );
  true
  gap> G := SymmetricGroup( 4 );; StructureDescription( G );; G;
  S4
  gap> StructureDescription( SerreFactorGroup( G ) );
  "1"
  gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
  <algebra-with-one over GF(2), with 2 generators>
  gap> DimensionOfUnitaryGroup( kG );
  16
  gap> k := HomalgRingOfIntegersInSingular( 2 );
  GF(2)
  gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

There are no non-principal blocks, so we only consider the principal block:

```
                            ──────── Example ────────
  gap> B0 := PrincipalBlock( kG );
  <two-sided ideal in <algebra-with-one over GF(2), with 2 generators>,
    (dimension 24)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
                            ──────── Example ────────
  gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
  <A torsion-free ideal given by 3 generators>
  gap> AffineDimension( L0 );
  7
  gap> Display( L0 );
  x1+x3+x5,
  x1+x3+x7+x9,
  x9

  An ideal generated by the 3 entries of the above matrix
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong 1 \cong G/G^\circ$:

```
                            ──────── Example ────────
  gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
  <A torsion-free ideal given by 3 generators>
  gap> AffineDimension( M0 );
  6
  gap> Display( M0 );
  y5+y6,
  y2+y6,
  y1+y4+y6

  An ideal generated by the 3 entries of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0 / \mathfrak{r}_0)$ is connected:

```
                          ─── Example ───
  gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
  <A torsion-free ideal given by 3 generators>
  gap> AffineDimension( I0 );
  3
  gap> Dec0 := RadicalDecomposition( I0 );
  [ <A torsion-free ideal given by 2 generators> ]
  gap> AffineDegree( Dec0[1] );
  2
  gap> Perform( Dec0, Display );
  y1+y2+y3+y4+y5+1,
  y2^2+y2*y3+y2*y4+y3*y4+y2*y5+y3*y5+y4*y5+y5^2+y2+y5


  An ideal generated by the 2 entries of the above matrix
```

2.10.6 SL(2,3)

The following computations show that $E_{SL_2(\mathbb{F}_3)}$ is trivial. This is a group for which the connected $U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)$ is of degree $4 > 1$.

```
                          ─── Example ───
  gap> LoadPackage( "Blocks" );
  true
  gap> G := SmallGroup( 24, 3 );; StructureDescription( G );; G;
  SL(2,3)
  gap> StructureDescription( SerreFactorGroup( G ) );
  "1"
  gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
  <algebra-with-one over GF(2), with 4 generators>
  gap> DimensionOfUnitaryGroup( kG );
  12
  gap> k := HomalgRingOfIntegersInSingular( 2 );
  GF(2)
  gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

There are no non-principal blocks, so we only consider the principal block:

```
                          ─── Example ───
  gap> B0 := PrincipalBlock( kG );
  <two-sided ideal in <algebra-with-one over GF(2), with 4 generators>,
   (dimension 24)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i/\mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
                          ─── Example ───
  gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
  <A torsion-free ideal given by 9 generators>
  gap> AffineDimension( L0 );
  9
  gap> AffineDegree( L0 );
  2
  gap> IsLowerPartExtensionOfAffineSpaces( B0 );
  true
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong 1 \cong G/G^\circ$ (however, we see that $U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)$ is of degree $4 > 1$):

```
―――――――――――――――――――― Example ――――――――――――――――――――
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 4 generators>
gap> RadicalDecomposition( M0 );
[ <A torsion-free ideal given by 4 generators> ]
gap> AffineDimension( M0 );
2
gap> AffineDegree( M0 );
4
gap> Display( M0 );
y4+y5+y6,
y2+y6,
y3^2+y5^2+y6^2+y3,
y1^2+y5^2+y1

An ideal generated by the 4 entries of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
―――――――――――――――――――― Example ――――――――――――――――――――
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 3 generators>
gap> AffineDimension( I0 );
1
gap> Dec0 := RadicalDecomposition( I0 );
[ <A torsion-free ideal given by 2 generators> ]
gap> AffineDegree( Dec0[1] );
2
gap> Perform( Dec0, Display );
y1+y2+y3+1,
y2^2+y2*y3+y3^2+y2+y3

An ideal generated by the 2 entries of the above matrix
```

2.11   Order 48

2.11.1   A4 : C4

```
―――――――――――――――――――― Example ――――――――――――――――――――
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 48, 30 );; StructureDescription( G );; G;
A4 : C4
gap> StructureDescription( SerreFactorGroup( G ) );
"C2"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 5 generators>
gap> DimensionOfUnitaryGroup( kG );
27
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

There are no non-principal blocks, so we only consider the principal block:

```
————————————————————— Example —————————————————————
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 5 generators>,
  (dimension 48)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
————————————————————— Example —————————————————————
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 16 generators>
gap> AffineDimension( L0 );
18
gap> AffineDegree( L0 );
10
gap> IsLowerPartExtensionOfAffineSpaces( B0 );
true
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$.

```
————————————————————— Example —————————————————————
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 5 generators>
gap> AffineDimension( M0 );
8
gap> m0 := RadicalDecomposition( M0 );
[ <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators> ]
gap> Perform( m0, Display );
y6+y7+y9+y12,
y4,
y3+y5+y8,
y2+y7+y8+y12,
y1+y8+y9

An ideal generated by the 5 entries of the above matrix
y6+y7+y9+y12,
y4+1,
y3+y5+y8,
y2+y7+y8+y12,
y1+y8+y9

An ideal generated by the 5 entries of the above matrix
```

An example where $U(B_0, B_0 / \mathfrak{r}_0)$ is *not* connected:

```
————————————————————— Example —————————————————————
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 12 generators>
gap> AffineDimension( I0 );
1
gap> Dec0 := RadicalDecomposition( I0 );
[ <A torsion-free ideal given by 4 generators>,
  <A torsion-free ideal given by 4 generators> ]
gap> Perform( Dec0, Display );
```

```
    y3+y5,
    y2+y4+y5+1,
    y1+y5,
    y4^2+y4*y5+y5^2+y4


    An ideal generated by the 4 entries of the above matrix
    y4,
    y2,
    y1+y3+y5+1,
    y3^2+y3*y5+y5^2+y3+y5


    An ideal generated by the 4 entries of the above matrix
```

Let *IG* be the augmentation ideal of *G* and $\mathfrak{a} := IG^2 \cap \mathfrak{r}_0$. We now prove that $U(B_0, B_0/\mathfrak{a})$ has 4 components.

─────────────── Example ───────────────

```
gap> a := Intersection( AugmentationIdealPowers( kG ).2, RadicalOfAlgebra( B0 ) );
<algebra of dimension 42 over GF(2)>
gap> filt := InducedFiltration( k, [ B0, a ] );
<An ascending filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A free left module of rank 6 on free generators>
  -1:   <A free left module of rank 42 on free generators>
of
<A free left module of rank 48 on free generators>>
gap> phi := DefiningMorphismOfUnitaryGroup( filt );
<A homomorphism of rings>
gap> I := IdealContainedInKernelViaEliminateOverBaseRing( phi );
<A torsion-free ideal given by 5 generators>
gap> Display( I );
y3*y4+y1*y5+y2*y6,
y1*y3+y2*y4+y2*y5+y4*y5+y1*y6+y3*y6,
y1*y2+y3*y5+y4*y6,
y2*y3+y1*y4+y5*y6,
y1^2+y2^2+y3^2+y4^2+y5^2+y6^2+1


An ideal generated by the 5 entries of the above matrix
gap> i := RadicalDecomposition( I );
[ <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators> ]
gap> Perform( i, Display );
y5+1,
y4+1,
y2+1,
y1+y3+y6,
y3^2+y3*y6+y6^2+1


An ideal generated by the 5 entries of the above matrix
y6+1,
y3+1,
y2+y4+y5,
y1+1,
```

```
y4^2+y4*y5+y5^2+1

An ideal generated by the 5 entries of the above matrix
y5,
y4,
y2,
y1+y3+y6+1,
y3^2+y3*y6+y6^2+y3+y6

An ideal generated by the 5 entries of the above matrix
y6,
y3,
y2+y4+y5+1,
y1,
y4^2+y4*y5+y5^2+y4+y5

An ideal generated by the 5 entries of the above matrix
gap> pim1 := Pullback( phi, AMaximalIdealContaining( i[1] ) );
<A torsion-free ideal given by 6 generators>
gap> pim2 := Pullback( phi, AMaximalIdealContaining( i[2] ) );
<A torsion-free ideal given by 6 generators>
gap> pim3 := Pullback( phi, AMaximalIdealContaining( i[3] ) );
<A torsion-free ideal given by 6 generators>
gap> pim4 := Pullback( phi, AMaximalIdealContaining( i[4] ) );
<A torsion-free ideal given by 6 generators>
```

2.11.2 2.S4

```
————————————— Example —————————————
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 48, 28 );; StructureDescription( G );; G;
C2 . S4 = SL(2,3) . C2
gap> StructureDescription( SerreFactorGroup( G ) );
"C2"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 5 generators>
gap> DimensionOfUnitaryGroup( kG );
24
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

There are no non-principal blocks, so we only consider the principal block:

```
————————————— Example —————————————
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 5 generators>,
  (dimension 48)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
————————————— Example —————————————
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 27 generators>
```

```
gap> AffineDimension( L0 );
20
gap> AffineDegree( L0 );
18
gap> IsLowerPartExtensionOfAffineSpaces( B0 );
true
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$.

```
                           ── Example ──
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 6 generators>
gap> AffineDimension( M0 );
3
gap> m0 := RadicalDecomposition( M0 );
[ <A torsion-free ideal given by 6 generators>,
  <A torsion-free ideal given by 6 generators> ]
gap> Perform( m0, Display );
y4+y5,
y3+y6+y8,
y2+y6+y7,
y1+y5+y6+y8,
y7^2+y8^2+y9^2+y5+y6+y8,
y6^2+y9^2+y5+y8

An ideal generated by the 6 entries of the above matrix
y4+y5+1,
y3+y6+y8,
y2+y6+y7,
y1+y5+y6+y8+1,
y7^2+y8^2+y9^2+y5+y6+y8+1,
y6^2+y9^2+y5+y8+1

An ideal generated by the 6 entries of the above matrix
```

An example where $U(B_0, B_0/\mathfrak{r}_0)$ is *not* connected:

```
                           ── Example ──
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 12 generators>
gap> AffineDimension( I0 );
1
gap> Dec0 := RadicalDecomposition( I0 );
[ <A torsion-free ideal given by 4 generators>,
  <A torsion-free ideal given by 4 generators> ]
gap> Perform( Dec0, Display );
y3+y5,
y2+y4+y5+1,
y1+y5,
y4^2+y4*y5+y5^2+y4

An ideal generated by the 4 entries of the above matrix
y4,
y2,
```

```
   y1+y3+y5+1,
   y3^2+y3*y5+y5^2+y3+y5


An ideal generated by the 4 entries of the above matrix
```

Let *IG* be the augmentation ideal of *G* and $\mathfrak{a} := IG^2 \cap \mathfrak{r}_0$. We now prove that $U(B_0, B_0/\mathfrak{a})$ has 4 components.

```
_____ Example _____
gap> a := Intersection( AugmentationIdealPowers( kG ).2, RadicalOfAlgebra( B0 ) );
<algebra of dimension 42 over GF(2)>
gap> filt := InducedFiltration( k, [ B0, a ] );
<An ascending filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A free left module of rank 6 on free generators>
  -1:   <A free left module of rank 42 on free generators>
of
<A free left module of rank 48 on free generators>>
gap> phi := DefiningMorphismOfUnitaryGroup( filt );
<A homomorphism of rings>
gap> I := IdealContainedInKernelViaEliminateOverBaseRing( phi );
<A torsion-free ideal given by 5 generators>
gap> Display( I );
y3*y4+y1*y5+y2*y6,
y1*y3+y2*y4+y2*y5+y4*y5+y1*y6+y3*y6,
y1*y2+y3*y5+y4*y6,
y2*y3+y1*y4+y5*y6,
y1^2+y2^2+y3^2+y4^2+y5^2+y6^2+1


An ideal generated by the 5 entries of the above matrix
gap> i := RadicalDecomposition( I );
[ <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators>,
  <A torsion-free ideal given by 5 generators> ]
gap> Perform( i, Display );
y5+1,
y4+1,
y2+1,
y1+y3+y6,
y3^2+y3*y6+y6^2+1


An ideal generated by the 5 entries of the above matrix
y6+1,
y3+1,
y2+y4+y5,
y1+1,
y4^2+y4*y5+y5^2+1


An ideal generated by the 5 entries of the above matrix
y5,
y4,
y2,
y1+y3+y6+1,
y3^2+y3*y6+y6^2+y3+y6
```

```
An ideal generated by the 5 entries of the above matrix
y6,
y3,
y2+y4+y5+1,
y1,
y4^2+y4*y5+y5^2+y4+y5

An ideal generated by the 5 entries of the above matrix
gap> pim1 := Pullback( phi, AMaximalIdealContaining( i[1] ) );
<A torsion-free ideal given by 6 generators>
gap> pim2 := Pullback( phi, AMaximalIdealContaining( i[2] ) );
<A torsion-free ideal given by 6 generators>
gap> pim3 := Pullback( phi, AMaximalIdealContaining( i[3] ) );
<A torsion-free ideal given by 6 generators>
gap> pim4 := Pullback( phi, AMaximalIdealContaining( i[4] ) );
<A torsion-free ideal given by 6 generators>
```

2.11.3 GL(2,3)

The following computations show that $E_{GL(2,3)}$ is trivial.

———— Example ————
```
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 48, 29 );; StructureDescription( G );; G;
GL(2,3)
gap> StructureDescription( SerreFactorGroup( G ) );
"1"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 5 generators>
gap> DimensionOfUnitaryGroup( kG );
30
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

There are no non-principal blocks, so we only consider the principal block:

———— Example ————
```
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 5 generators>,
  (dimension 48)>
```

Verify statement 1 for $j = 0$, i.e., $U\big(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1}\big)$ is connected for all $i \geq 2$:

———— Example ————
```
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 17 generators>
gap> AffineDimension( L0 );
23
gap> AffineDegree( L0 );
16
gap> IsLowerPartExtensionOfAffineSpaces( B0 );
true
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$.

```
────────────────────────── Example ──────────────────────────
 gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
 <A torsion-free ideal given by 5 generators>
 gap> AffineDimension( M0 );
 4
 gap> RadicalDecomposition( M0 );
 [ <A torsion-free ideal given by 5 generators> ]
 gap> Display( M0 );
 y3+y6+y8,
 y2+y6+y7,
 y1+y4+y6+y8,
 y7^2+y8^2+y9^2+y4+y6+y8,
 y4^2+y5^2+y6^2+y9^2+y5+y8


 An ideal generated by the 5 entries of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0/\mathfrak{r}_0)$ is connected:

```
────────────────────────── Example ──────────────────────────
 gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
 <A torsion-free ideal given by 3 generators>
 gap> AffineDimension( I0 );
 3
 gap> Dec0 := RadicalDecomposition( I0 );
 [ <A torsion-free ideal given by 2 generators> ]
 gap> Display( Dec0[1] );
 y1+y2+y3+y4+y5+1,
 y2*y3+y3^2+y2*y4+y3*y4+y2*y5+y3*y5+y4*y5+y5^2+y3+y5


 An ideal generated by the 2 entries of the above matrix
```

2.12   Order 60

2.12.1   C15 : C4

The faithful semi-direct product $G := C_{15} \rtimes C_4$ is the unique group G of order less or equal to 60 having *trivial* center and nontrivial E_G. It is one of 4 special groups of order 60. The other nonfaithful semi-direct product $C_{15} \rtimes C_4$ is also a special group.

```
────────────────────────── Example ──────────────────────────
 gap> LoadPackage( "Blocks" );
 true
 gap> G := SmallGroup( 60, 7 );; StructureDescription( G );; G;
 C15 : C4
 gap> StructureDescription( SerreFactorGroup( G ) );
 "C2"
 gap> IsTrivial( Center( G ) );
 true
 gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
 <algebra-with-one over GF(2), with 4 generators>
 gap> DimensionOfUnitaryGroup( kG );
 32
```

```
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:
```
                        _____ Example _____
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 4 generators>,
  (dimension 4)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:
```
                        _____ Example _____
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A zero ideal>
gap> AffineDimension( L0 );
2
```

Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong C_2 \cong G/G^\circ$:
```
                        _____ Example _____
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M0 );
0
gap> m0 := RadicalDecomposition( M0 );
[ <A principal torsion-free ideal given by a cyclic generator>,
  <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( m0, Display );
y1+1

An ideal generated by the entry of the above matrix
y1

An ideal generated by the entry of the above matrix
```

Verify statement 2.(b), i.e., $U(B_0, B_0 / \mathfrak{r}_0)$ is connected:
```
                        _____ Example _____
gap> I0 := DefiningIdealOfUpperPartOfUnitaryGroup( B0 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( I0 );
0
gap> Dec0 := RadicalDecomposition( I0 );
[ <A principal torsion-free ideal given by a cyclic generator> ]
gap> Perform( Dec0, Display );
y1+1

An ideal generated by the entry of the above matrix
```

Now we consider the non-principal block B_1:
```
                        _____ Example _____
gap> B := RealNonPrincipalBlocksOfGroupAlgebra( kG );
[ <two-sided ideal in <algebra-with-one of dimension 60 over GF(2)>,
```

```
     (dimension 8)>,
   <two-sided ideal in <algebra-with-one of dimension 60 over GF(2)>,
     (dimension 16)> ]
gap> B1 := B[1];
<two-sided ideal in <algebra-with-one of dimension 60 over GF(2)>,
 (dimension 8)>
gap> IsSpecial( B1 );
true
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i / \mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
                    ─────────── Example ───────────
gap> DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
<A zero vector subspace>
```

Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1 / \mathfrak{r}_1^2)$ is connected:

```
                    ─────────── Example ───────────
gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
<A principal torsion-free ideal given by a cyclic generator>
gap> AffineDimension( M1 );
3
gap> Display( M1 );
y1

An ideal generated by the entry of the above matrix
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^n \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
                    ─────────── Example ───────────
gap> I1 := DefiningIdealOfUpperPartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 7 generators>
gap> AffineDimension( I1 );
1
gap> Dec1 := RadicalDecomposition( I1 );
[ <A torsion-free ideal given by 3 generators>,
  <A torsion-free ideal given by 3 generators> ]
gap> Perform( Dec1, Display );
y4,
y2,
y1^2+y1*y3+y3^2+1

An ideal generated by the 3 entries of the above matrix
y3,
y1,
y2^2+y2*y4+y4^2+1

An ideal generated by the 3 entries of the above matrix
```

Now we consider the non-principal block B_2:

```
                    ─────────── Example ───────────
gap> B2 := B[2];
<two-sided ideal in <algebra-with-one of dimension 60 over GF(2)>,
 (dimension 16)>
gap> IsSpecial( B2 );
false
```

Verify statement 1 for $j = 2$, i.e., $U(B_2, \mathfrak{r}_2^i / \mathfrak{r}_2^{i+1})$ is connected for all $i \geq 2$:

```
———————————————————— Example ————————————————————
gap> DefiningIdealOfLowerPartOfUnitaryGroup( B2 );
<A zero vector subspace>
```

Verify statement 3, i.e., $U(B_2, \mathfrak{r}_2 / \mathfrak{r}_2^2)$ is connected:

```
———————————————————— Example ————————————————————
gap> DefiningIdealOfMiddlePartOfUnitaryGroup( B2 );
<A zero vector subspace>
```

Verify the formula for E_G, i.e., $E_G \cong \pi_0(U(A, A/\mathfrak{r})) \cong \prod_{j=1}^n \pi_0(U(B_j, B_j/\mathfrak{r}_j)) \cong C_2$:

```
———————————————————— Example ————————————————————
gap> I2 := DefiningIdealOfUpperPartOfUnitaryGroup( B2 );
<A torsion-free ideal given by 24 generators>
gap> AffineDimension( I2 );
10
```

The remaining computation was done using MAGMA for the radical decomposition. It took MAGMA 2.20-9 about 47 minutes and 300MB to compute the decomposition.

```
———————————————————— Example ————————————————————
gap> Dec2 := RadicalDecomposition( I2 );
[ <A torsion-free ideal given by 24 generators> ]
```

2.12.2   A5

The following computations show that E_{A_5} is trivial. Serre showed that for $A = \overline{\mathbb{F}}_2 A_5$ with radical $\mathfrak{r}$ the factor group $U(A, A/\mathfrak{r})$ is connected. The 16-dimensional non-principal block B_1 is semi-simple. Hence, we only need to show that for the 44-dimensional principal block B_0 with radical $\mathfrak{r}_0$ the subgroup $U(B_0, \mathfrak{r}_0^2)$ (lower part) and the subfactor group $U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)$ (middle part) are connected:

```
———————————————————— Example ————————————————————
gap> LoadPackage( "Blocks" );
true
gap> G := AlternatingGroup( 5 );; StructureDescription( G );; G;
A5
gap> StructureDescription( SerreFactorGroup( G ) );
"1"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 2 generators>
gap> DimensionOfUnitaryGroup( kG );
37
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

As we mentioned above, we only consider the principal block:

```
———————————————————— Example ————————————————————
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 2 generators>,
  (dimension 44)>
```

Verify statement 1 for $j = 0$, i.e., $U(B_0, \mathfrak{r}_0^i / \mathfrak{r}_0^{i+1})$ is connected for all $i \geq 2$:

```
——————————————— Example ———————————————
gap> L0 := DefiningIdealOfLowerPartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 20 generators>
gap> AffineDimension( L0 );
17
gap> AffineDegree( L0 );
2
gap> IsLowerPartExtensionOfAffineSpaces( B0 );
true
```

The defininig ideal of the subgroup $U(B_0, \mathfrak{r}_0)$ is generated by 25 elements in 35 indeterminates. To compute the defining ideal `M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 )` of the middle part $U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)$ we need to eliminate (the last) 27 indeterminates. Verify statement 2, i.e., $\pi_0(U(B_0, \mathfrak{r}_0 / \mathfrak{r}_0^2)) \cong 1 \cong G/G^\circ$.

```
——————————————— Example ———————————————
gap> M0 := DefiningIdealOfMiddlePartOfUnitaryGroup( B0 );
<A torsion-free ideal given by 4 generators>
gap> AffineDimension( M0 );
4
gap> Display( M0 );
y5+y6+y8,
y4,
y2+y3+y7,
y1+y3+y6+y7+y8


An ideal generated by the 4 entries of the above matrix
```

2.13   Order 120

2.13.1   2.A5

The following computations only deal with the non-principal 2-block B_1 of $SL(2,5)$ and prove that $\pi_0(U(B_1)) \cong \pi_0(U(B_1, B_1/\mathfrak{r}_1))$. J.-P. Serre told me that he now has an a priori proof that $\pi_0(U(B_1)) \cong C_2$.

```
——————————————— Example ———————————————
gap> LoadPackage( "Blocks" );
true
gap> G := SmallGroup( 120, 5 );; StructureDescription( G );; G;
SL(2,5)
gap> StructureDescription( SerreFactorGroup( G ) );
"1"
gap> kG := GroupRingOverInvolutionSplittingField( 2, G );
<algebra-with-one over GF(2), with 2 generators>
gap> DimensionOfUnitaryGroup( kG );
60
gap> k := HomalgRingOfIntegersInSingular( 2 );
GF(2)
gap> SetCoefficientsRingForPolynomialAlgebra( kG, k );
```

We first consider the principal block:

```
─────────────────────── Example ───────────────────────
gap> B0 := PrincipalBlock( kG );
<two-sided ideal in <algebra-with-one over GF(2), with 2 generators>,
 (dimension 88)>
```

Now we consider the non-principal block B_1:

```
─────────────────────── Example ───────────────────────
gap> e := CentralNonPrincipalIdempotentsOfGroupAlgebra( kG );;
gap> B1 := BlockOfIdempotent( e[1] );
<two-sided ideal in <algebra-with-one over GF(2), with 2 generators>,
 (1 generators)>
gap> Dimension( B1 );
32
```

Verify statement 1 for $j = 1$, i.e., $U(B_1, \mathfrak{r}_1^i/\mathfrak{r}_1^{i+1})$ is connected for all $i \geq 2$:

```
─────────────────────── Example ───────────────────────
gap> DefiningIdealOfLowerPartOfUnitaryGroup( B1 );
<A zero vector subspace>
```

The following computation took GAP about 22min on a modern computer. Verify statement 3, i.e., $U(B_1, \mathfrak{r}_1/\mathfrak{r}_1^2)$ is connected:

```
─────────────────────── Example ───────────────────────
gap> M1 := DefiningIdealOfMiddlePartOfUnitaryGroup( B1 );
<A torsion-free ideal given by 6 generators>
gap> AffineDimension( M1 );
10
gap> Display( M1 );
y13+y14+y15,
y9+y12+y16,
y7+y8+y14,
y6,
y4,
y1

An ideal generated by the 6 entries of the above matrix
```

Chapter 3

Appendix

3.1 Summary of changes between successive versions

This section lists a summary of changes made to this document from version to version in reversed chronological order. The current version is indicated by the date on the front page. This section will disappear when this document reaches a final form.

Changes between:

- 18.02 – 20.07.2015

    - Updated output as produced by Blocks v2015.07.20.

- 18.01.2014 – 08.02.2015

    - Updated output as produced by Singular 4.0.1.
    - Restructured chapters.
    - Explicitly check the property `IsSpecial` for table blocks.
    - Use Blocks v2015.01.13 and update timings (it replaced `KernelSubobject` by the faster `IdealContainedInKernelViaEliminateOverBaseRing`).
    - Turn the `Log`-sections into testable `Example`-sections as computations are becoming more feasible.
    - Updated examples and fixed typos.
    - Corrected and enhanced some comments, e.g., $E_{SL(2,3)}$ is now proven trivial.
    - Added S_3 to the new Section 2.7.
    - Added the three groups $A_4 \rtimes C_4$, $2.S_4$, and $GL(2,3)$ to the new Section 2.11.
    - Used MAGMA to compute the missing radical decomposition for $C_{15} \rtimes C_4$ in Subsection 2.12.1.

- 31.12 – 18.01.2014

    - Updated timings of computations.

- 22.11 – 31.12.2013

    - Fixed a typo in Subsection 2.10.3: C_4 should be $C_2 \times C_2$.

39

- – Use `GroupRingOverInvolutionSplittingField( 2, G )` and updated output to comply with new version of the package Blocks (v2013.12.27).
- – Added the *faithful* semi-direct product $C_{15} \rtimes C_4$ as the new Subsection 2.12.1.

- 13.11 – 22.11.2013

    - – Update output to comply with new version of the package Blocks (v2013.11.08).
    - – Updated BibTeX entry for [Ser05].
    - – Updated URL of homalg-online.

- 04.11 – 13.11.2013

    - – Added $C_2 \times A_4$ as the new Subsection 2.10.4.
    - – Added a link to the homalg online server a the end of the Introduction (Chapter 1), where all computations performed here can be reproduced an verified.
    - – Added a preliminary treatment of $SL(2,5) = 2.A_5$ as the new Subsection 2.13.1.
    - – Cite [Ser14] in the Introduction (Chapter 1).

- 24.10 – 04.11.2013

    - – Added reference to the software in the Introduction (Chapter 1.1).
    - – Modified the notation φ_J to $\varphi_{A/J}$ to be compatible with the "double" index notation $\varphi_{I/J}$ in Section 2.1.
    - – Defined what I call upper, middle, and lower part of a unitary group in Section 2.1.
    - – Described the algorithm to compute the image in Section 2.2.
    - – Added reference to [Ser05] in Section 2.3.
    - – Added the discussion of Bovdi's and Rosa's paper [BR00] as the new Section 2.4. This was the content of my email on the 26th of August (fixed reported typo).
    - – I mistakenly programmed `DefiningIdealOfLowerPartOfUnitaryGroup` in the package Blocks to compute the defining ideal of $U(B, \mathfrak{r}^2/\mathfrak{r}^3)$ instead of $U(B, \mathfrak{r}^2)$. Once I added the rule that the sum of dimensions of all involved subfactors should add up to the `DimensionOfUnitaryGroup( kG )` the computer objected. This is now fixed. (In the computer algebra software we develop we created tools to combine algorithms with theoretical results. We refer to the latter as zero-cost algorithms or $O(1)$-algorithms [BLH12]).
    - – In my email on the 26th of August I wrote that for $G = C_4 \rtimes C_3$ the degree of the factor group $U(kG, kG/\mathfrak{r})$ is 8. There I was mistakenly referring to the non-reduced structure. The correct degree of $U(kG, kG/\mathfrak{r})$ is 4 (see the end of Subsection 2.8.1).
    - – Added the non-faithful semi-direct product $C_5 \rtimes C_4$ to the new Section 2.9.
    - – Added S_4 as the new Subsection 2.10.5.
    - – Added $SL(2,3) = SL_2(\mathbb{F}_3)$ as the new Subsection 2.10.6. The algorithms did not yet succeed to compute $E_{SL_2(\mathbb{F}_3)}$, however, this is a group for which the connected subfactor $U(B_0, \mathfrak{r}_0/\mathfrak{r}_0^2)$ is of degree $2 > 1$.
    - – Added A_5 to the new Section 2.12.

3.2 About this document

The printed output of the computations in all examples displayed in this document has been computed once and is now hard-coded in the example files (after being checked for plausibility). This is done on purpose to guarantee that future changes to the program cannot introduce mistakes which alter the results. In particular, the computer is able to check if the printed output coincides with the recomputed one, which is among the few possible automatic integrity checks computer algebra can currently provide.

The GAP4 programming language (like all other programming languages in computer algebra which I am aware of) does not have the linguistic capabilities of a proof assistant (e.g., Coq). A proof assistant enables the programmer to formulate assumptions about his programs (in form of meta statements) and provides him with tools to perform correctness proofs of these assumptions. Such a capability is still missing in computer algebra systems for several reasons. All proof assistants rely on a strict functional programming paradigm. In fact, a proof assistant can indeed serve as a computer algebra system. However, in my opinion, the functional paradigm is extremely impractical for everyday computer algebra, which rather relies on other programming paradigms, e.g., those allowing side effects. When Donald Knuth was asked about functional programming during his Turing Lecture in Glasgow he replied: "With one or even two hands tied behind your back it's hard to do anything dangerous." (http://homepages.cs.ncl.ac.uk/m.j.bell1/blog/?p=49). Much more research and human resources are needed to bring proof assistants to everyday computer algebra.

The computer algebra system Singular [DGPS14] is used as an oracle for Gröbner basis computations, which are the most expensive computational tasks we have to perform. All given timings above mainly depend on the implemented algorithms in the current version of the package Blocks (see date on the title page), and, hence, also on the current versions of the yet single threaded computer algebra systems GAP4 and Singular.

This human readable document has been generated out of several GAP readable files by the procedure AutoDocWorksheet in the GAPDoc-based package AutoDoc. In particular, the code used to produce this document can be used to check the integrity of the used software.

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